MODEL SOLUTIONS

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Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Wednesday 19 June 2024

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/3A**

Further Mathematics

Advanced

PAPER 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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1. (a) Given that

$$y = \ln\left(3 + x^2\right)$$

complete the table with the value of y corresponding to $x = 3$, giving your answer to 4 significant figures.

x	2	2.5	3	3.5	4	4.5	5
y	1.946	2.225	2.485	2.725	2.944	3.146	3.332

1

(1)

In part (b) you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(b) Use Simpson's rule with all the values of y in the completed table to estimate, to 3 significant figures, the value of

$$\int_2^5 \ln\left(3 + x^2\right) \mathrm{d}x$$

(3)

(c) Using your answer to part (b) and making your method clear, estimate the value of

$$\int_2^5 \ln\sqrt{\left(3 + x^2\right)}\, \mathrm{d}x$$

(1)

(b) Using 6 steps to evaluate the integral between 2 and 5

$h = \dfrac{5-2}{6} = 0.5$ 1

Simpson's Rule : $\int_a^b f(x)\,dx = \frac{1}{3}h\left[y_0 + y_n + 4(y_1 + y_3) + 2(y_2 + y_4 \dots)\right]$

$\int_2^5 \ln(3+x^2)\,dx = \frac{1}{3} \times 0.5 \times \left[1.946 + 3.332 + 4(2.225 + 2.725 + 3.146) + 2(2.485 + 2.944)\right]$ 1

$\int_2^5 \ln(3+x^2)\,dx = \frac{1}{6} \times 48.52 = 8.09$ 1

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Question 1 continued

(c) $0.5 \times 8.09 = 4.045$ 1

$\ln x^a = a \ln x$ and $\sqrt{x} = x^{\frac{1}{2}}$ so $\ln \sqrt{x} = \frac{1}{2} \ln x$

$\therefore$ multiply estimate by $\frac{1}{2}$ (0.5)

(Total for Question 1 is 5 marks)

2. Use algebra to determine the values of x for which

$$|x^2 - 2x| \leqslant x$$

(4)

$|x^2 - 2x| \leq x$

$x^2 - 2x \leq x$ AND $x^2 - 2x \leq -x$ ①

$x^2 - 3x \leq 0$ $x^2 - x \leq 0$

$x(x-3) = 0$ $x(x-1) = 0$

$\therefore x = 0, 3$ ① $\therefore x = 0, 1$ ①

If $x < 0$: $|-1^2 - 2\times -1| \leq -1 \Rightarrow 3 \leq -1$ ✗

If $0 \leq x \leq 1$. $|\frac{1}{2}^2 - 2\times\frac{1}{2}| \leq \frac{1}{2} \Rightarrow \frac{3}{4} \leq \frac{1}{2}$ ✗

If $1 \leq x \leq 3$. $|\frac{3}{2}^2 - 2\times\frac{3}{2}| \leq \frac{3}{2} \Rightarrow \frac{3}{4} \leq \frac{3}{2}$ ✓

If $x > 3$. $|4^2 - 2\times 4| \leq 4 \Rightarrow 8 \leq 4$ ✗

$\therefore x = 0, \quad 1 \leq x \leq 3$ ①

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Question 2 continued

(Total for Question 2 is 4 marks)

3. Use L'Hospital's rule to show that

$$\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0$$

(6)

L'Hospital's Rule. $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a}\left(\frac{f'(x)}{g'(x)}\right)$

$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$ 1 ← put into form $\frac{f(x)}{g(x)}$

$f(x) = x - \sin x \implies f'(x) = 1 - \cos x$

$g(x) = x \sin x \implies u = x \quad v = \sin x$

$u' = 1 \quad v' = \cos x$

$g(x) = uv' + vu'$

$g(x) = x\cos x + \sin x$

$\lim_{x \to 0} \frac{1 - \cos x}{x\cos x + \sin x}$ 1 1 is still indeterminate $0 \times \cos 0 + \sin 0 = 0$

So differentiate again

$f''(x) = \sin x$

$g''(x) = uv' + vu' + \cos x$

$u = x \quad v = \cos x$

$u' = 1 \quad v' = -\sin x$

$g''(x) = \cos x - x\sin x + \cos x$

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Question 3 continued

$$\lim_{x \to 0} \frac{f''(x)}{g''(x)} = \lim_{x \to 0} \frac{\sin x}{2\cos x - x\sin x}$$ (1) (1)

As $x \to 0$, $\frac{\sin(0)}{2\cos(0) - 0\sin(0)}$ (1) $= \frac{0}{2 \times 0 - 0 \times 0} = 0$ (1)

(Total for Question 3 is 6 marks)

4.

[*The Taylor series expansion of* $\mathrm{f}(x)$ *about* $x = a$ *is given by*

$$\mathrm{f}(x) = \mathrm{f}(a) + (x-a)\mathrm{f}'(a) + \frac{(x-a)^2}{2!}\mathrm{f}''(a) + \ldots + \frac{(x-a)^r}{r!}\mathrm{f}^{(r)}(a) + \ldots$$]

The curve with equation $y = \mathrm{f}(x)$ satisfies the differential equation

$$\cos x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + \sin x = 0$$

Given that $\left(\frac{\pi}{4}, 1\right)$ is a stationary point of the curve,

(a) determine the nature of this stationary point, giving a reason for your answer.

(2)

(b) Show that $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \sqrt{2} - 2$ at this stationary point.

(4)

(c) Hence determine a series solution for y, in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in simplest form.

(2)

(a) $\cos\left(\frac{\pi}{4}\right)\frac{d^2y}{dx^2} + (1)^2(0) + \sin\left(\frac{\pi}{4}\right) = 0$

$\frac{\sqrt{2}}{2} \times \frac{d^2y}{dx^2} + \frac{\sqrt{2}}{2} = 0$

$\frac{d^2y}{dx^2} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$\frac{d^2y}{dx^2} = -1$

$\frac{d^2y}{dx^2} < 0 \quad \therefore$ the point is a local maximum

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Question 4 continued

(b) $\cos x \frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + \sin x = 0$

Differentiate each term using the product rule.

$\cos x \frac{d^2y}{dx^2}$ $\quad u = \cos x \quad v = \frac{d^2y}{dx^2}$

$u' = -\sin x \quad v' = \frac{d^3y}{dx^3}$

$uv' + vu' = \cos x \frac{d^3y}{dx^3} - \sin x \frac{d^2y}{dx^2}$

$y^2 \frac{dy}{dx}$ $\quad u = y^2 \quad v = \frac{dy}{dx}$

$u' = 2y\frac{dy}{dx} \quad v' = \frac{d^2y}{dx^2}$

$uv' + vu' = y^2 \frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2$

$\cos x \frac{d^3y}{dx^3} - \sin x \frac{d^2y}{dx^2} + y^2\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 + \cos x = 0$ ①

$\frac{d^3y}{dx^3} = \left[(\sin x - y^2)\frac{d^2y}{dx^2} - 2y\left(\frac{dy}{dx}\right)^2 - \cos x\right] \div \cos x$ ①

①

$\frac{d^3y}{dx^3} = \left[(\sin\frac{\pi}{4} - 1^2) \times (-1) - 2(1)(0)^2 - \cos\frac{\pi}{4}\right] \div \cos\frac{\pi}{4}$

$\frac{dy}{dx} = 0$ because its a stationary point

$\frac{d^3y}{dx^3} = \left[\frac{2-\sqrt{2}}{2} - 0 - \frac{\sqrt{2}}{2}\right] \div \frac{\sqrt{2}}{2}$

$\frac{d^3y}{dx^3} = \left[1 - \sqrt{2}\right] \div \frac{\sqrt{2}}{2}$

$\frac{d^3y}{dx^3} = \sqrt{2} - 2$ ①

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Question 4 continued

(c) $y = 1 + \left(x - \frac{\pi}{4}\right)(0) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}(-1) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}(\sqrt{2} - 2)$ 1

$y = 1 - \frac{\left(x - \frac{\pi}{4}\right)^2}{2} + \frac{\left(x - \frac{\pi}{4}\right)^3(\sqrt{2} - 2)}{6} + \ldots$ 1

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Question 4 continued

(Total for Question 4 is 8 marks)

5. $$y = e^{3x} \sin x$$

(a) Use Leibniz's theorem to show that

$$\frac{d^4y}{dx^4} = 28e^{3x} \sin x + 96e^{3x} \cos x$$

(6)

(b) Hence express $\frac{d^4y}{dx^4}$ in the form

$$Re^{3x} \sin(x + \alpha)$$

where R and α are constants to be determined, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

(3)

(a) Leibnitz's Theorem $\frac{d^n y}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k u}{dx^k} \frac{d^{n-k} v}{dx^{n-k}}$

$y = e^{3x} \sin x$ $\quad \frac{d^4y}{dx^4}$ so differentiate to $u^{(4)}$

$u = e^{3x}$ ① $\qquad v = \sin x$ ①

$u' = 3e^{3x} \qquad v' = \cos x$

$u'' = 9e^{3x} \qquad v'' = -\sin x$

$u''' = 27e^{3x} \qquad v''' = -\cos x$

$u'''' = 81e^{3x}$ ① $\qquad v'''' = \sin x$ ①

$$\frac{d^4y}{dx^4} = (u \times v'''') + (4 \times u' \times v''') + (6 \times u'' \times v'') + (4 \times u''' \times v') + (u'''' \times v)$$

$$\frac{d^4y}{dx^4} = (e^{3x} \times \sin x) + (4 \times 3e^{3x} \times -\cos x) + (6 \times 9e^{3x} \times -\sin x) + (4 \times 27e^{3x} \times \cos x) + (81e^{3x} \times \sin x)$$ ①

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Question 5 continued

$$\frac{d^4y}{dx^4} = 28e^{3x}\sin x + 96e^{3x}\cos x$$ 1 combine $\sin x$ / $\cos x$ terms

(b) $R = \sqrt{a^2 + b^2}$ $\quad \tan\alpha = \pm\frac{b}{a}$

$R = \sqrt{28^2 + 96^2} = \sqrt{10\,000} = 100$ 1

$\tan\alpha = \frac{96}{28}$

$\alpha = \tan^{-1}\left(\frac{96}{28}\right) = 1\ 287$ 1

$\therefore \frac{d^4y}{dx^2} = 100e^{3x}\sin(x + 1\ 29)$ 1

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Question 5 continued

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Question 5 continued

(Total for Question 5 is 9 marks)

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Turn over

6. The ellipse E has equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive constants.

Given that

- the eccentricity of H is the reciprocal of the eccentricity of E
- the coordinates of the foci of H are the same as the coordinates of the foci of E

determine

(i) the value of a

(ii) the value of b

(6)

(i) Use equation to find e_E.

$\frac{x^2}{25^2} + \frac{y^2}{9^2} = 1 \implies 9 = 25(1 - e^2)$ 1

$1 - \frac{9}{25} = e^2$

for ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$b^2 = a^2(1 - e^2)$

$\sqrt{\frac{16}{25}} = e$

$\frac{4}{5} = e_E$

$5 \times \frac{4}{5} = 4$

Find reciprocal of e_E to get e_H

reciprocal $= \frac{1}{x} \longrightarrow e_H = \frac{1}{\frac{4}{5}} = \frac{5}{4}$ 1

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Question 6 continued

Use $e_H = \frac{5}{4}$ to find a_H (foci are the same).

$a_H e_H = 5 \times \frac{4}{5}$ (1) ← focus of ellipse $= ae$

$a_H = \dfrac{5 \times \frac{4}{5}}{\frac{5}{4}} = \dfrac{16}{5}$ (1)

(b) Use a_H and e_H with $b^2 = a^2(1-e^2)$ to find b_H

$b^2 = \left(\frac{16}{5}\right)^2 \times \left[1 - \left(\frac{5}{4}\right)^2\right]$ (1)

$b^2 = \pm \dfrac{144}{25}$

$b_H = \dfrac{12}{5}$ (1)

(Total for Question 6 is 6 marks)

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to show that

$$\int \frac{1}{2\sin\theta + \cos\theta + 2}\,d\theta = \int \frac{a}{(t+b)^2 + c}\,dt$$

where a, b and c are constants to be determined.

(3)

(b) Hence show that

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2\sin\theta + \cos\theta + 2}\,d\theta = \ln\left(\frac{2\sqrt{3}}{3}\right)$$

(4)

(a) $\sin\theta = \frac{2t}{1+t^2}$ $\quad \cos\theta = \frac{1-t^2}{1+t^2}$ using t-formulae

$\tan\frac{\theta}{2} = t \implies \theta = 2\tan^{-1}t \implies \frac{d\theta}{dt} = \frac{2}{1+t^2}$

in formula book $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$$\int \frac{1}{2\sin\theta + \cos\theta + 2}\,d\theta = \int \frac{1}{2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) + 2}\,\frac{2dt}{1+t^2} \quad 1$$

$$= \int \frac{2(\cancel{1+t^2})}{2\left(\frac{2t}{\cancel{1+t^2}}\right) + \left(\frac{1-t^2}{\cancel{1+t^2}}\right) + 2}\,dt$$

$$= \int \frac{2(1+t^2)}{4t + (1-t^2) + 2(1+t^2)}\,dt$$

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Question 7 continued

$$= \int \frac{2}{t^2 + 4t + 3} \, dt \quad (1)$$

$$= \int \frac{2}{(t+2)^2 - 1} \, dt \quad (1)$$

(b) from formula book $\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$

$$\int \frac{2}{(t+2)^2 - 1} \, dt = \frac{1}{2 \times 1} \ln \left| \frac{(t+2)-1}{(t+2)+1} \right| \quad (1)$$

To find limits use $t = \tan \frac{\theta}{2}$

$\frac{2\pi}{3}$. $t = \tan \frac{2\pi}{6} = \sqrt{3}$

$\frac{\pi}{2}$. $t = \tan \frac{\pi}{4} = 1 \quad (1)$

$$\ln \left| \frac{\sqrt{3}+1}{\sqrt{3}+3} \right| - \ln \left| \frac{1+1}{1+3} \right| = \ln \left| \frac{\sqrt{3}+1}{\sqrt{3}+3} - \frac{1}{2} \right| \quad (1)$$

rationalise the denominator $\rightarrow$ $= \ln \left| \frac{2\sqrt{3}+2}{\sqrt{3}+3} \times \frac{\sqrt{3}-3}{\sqrt{3}-3} \right|$

$$= \ln \left| \frac{2\sqrt{3}}{3} \right| \quad (1)$$

(Total for Question 7 is 7 marks)

8. The parabola P has equation $y^2 = 4ax$, where a is a positive constant.

The point $A\left(at^2, 2at\right)$, where $t \neq 0$, lies on P.

(a) Use calculus to show that an equation of the tangent to P at A is

$$yt = x + at^2$$

(3)

The point $B(2k^2, 4k)$ and the point $C(2k^2, -4k)$, where k is a constant, lie on P.

The tangent to P at B and the tangent to P at C intersect at the point D.

Given that the area of the triangle BCD is 432

(b) determine the coordinates of B and the coordinates of C.

(5)

(a) $y^2 = 4ax$

differentiate both sides w.r.t x

$2y\dfrac{dy}{dx} = 4a$

$\dfrac{dy}{dx} = \dfrac{4a}{2y}$

At A, $x = at^2$, $y = 2at$:

$\dfrac{dy}{dx} = \dfrac{4a}{2(2at)} = \dfrac{4a}{4at} = \dfrac{1}{t}$ 1

gradient of P at A

$y - 2at = \dfrac{1}{t}(x - at^2)$ 1 using $y - y_1 = m(x - x_1)$

$\times t$

$yt - 2at^2 = x - at^2$

$yt = x + at^2$ 1

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Question 8 continued

(b) Notice $B(2k^2, 4k)$ is $(at^2, 2at)$ with $a = 2$, $t = k$

Tangent at B: $yk = x + 2k^2$ ← using equation from (a)

Tangent at C: $-yk = x + 2k^2$ ① ← C is a reflection of B in the x-axis, so make y negative

Solve simultaneously:

① $yk = x + 2k^2$

② $-yk = x + 2k^2 \implies x = -2k^2 - yk$ ③

③ into ①: $yk = -2k^2 - yk + 2k^2$

$yk = -yk$

$y = -y$

$2y = 0$

$y = 0$

$y = 0$ into ③: $x = -2k^2 - yk$

$x = -2k^2 - (0)k$

$x = -2k^2$ ①

Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

Base $= 4k - -4k = 8k$

Height $= 2k^2 - -2k^2 = 4k^2$

Question 8 continued

$\frac{1}{2}(8k \times 4k^2) = 432$ ①

$32k^3 = 864$

$k^3 = 27$

$k = 3$ ①

$B(2(3^2), 4(3)) = B(18, 12)$

$C(2(3^2), -4(3)) = C(18, -12)$ ①

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Question 8 continued

(Total for Question 8 is 8 marks)

9. (i) The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

The line l_2 has equation $\mathbf{r} = \begin{pmatrix} 13 \\ 5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

where λ and μ are scalar parameters.

The lines l_1 and l_2 intersect at the point P.

(a) Determine the coordinates of P.

(2)

Given that the plane Π contains both l_1 and l_2

(b) determine a Cartesian equation for Π.

(4)

(ii) Determine a Cartesian equation for each of the two lines that

- pass through (0, 0, 0)
- make an angle of 60° with the x-axis
- make an angle of 45° with the y-axis

(4)

(i)(a) ① : $2 + 3\lambda = 13 + \mu$

② : $-3 + 4\lambda = 5 - 2\mu$

③ : $1 - \lambda = 8 + 5\mu$

① : $3\lambda = 11 + \mu$

$\lambda = \frac{1}{3}(11 + \mu)$

sub λ into ③ : $1 - \frac{1}{3}(11 + \mu) = 8 + 5\mu$

$3 - 11 - \mu = 24 + 15\mu$

$-32 = 16\mu$

$-2 = \mu$

$\lambda = \frac{1}{3}(11 + (-2)) = \frac{1}{3}(9) = 3$ ① for λ OR μ

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Question 9 continued

$$\begin{pmatrix}2\\-3\\1\end{pmatrix} + \underset{\lambda}{\underset{\uparrow}{3}}\begin{pmatrix}3\\4\\-1\end{pmatrix} = \begin{pmatrix}2+3(3)\\-3+3(4)\\1+3(-1)\end{pmatrix} = \begin{pmatrix}11\\9\\-2\end{pmatrix} \quad ①$$

(i)(b) Π $\quad r \cdot n = a \cdot n$ where a is a point, n is the normal to the plane

$$n = \begin{vmatrix} i & j & k \\ 3 & 4 & -1 \\ 1 & -2 & 5 \end{vmatrix} \begin{array}{l} \\ \leftarrow \text{direction of } l_1 \\ \leftarrow \text{direction of } l_2 \end{array}$$

$$n = [4\times5-(-1)\times(-2)]i - [3\times5-(-1)\times1]j + [3\times(-2)-4\times1]k$$

$$n = (20-2)i - (15+1)j + (-6-4)k \quad ①$$

$$n = 18i - 16j - 10k \quad ①$$

$$r \cdot \begin{pmatrix}18\\-16\\-10\end{pmatrix} = \begin{pmatrix}11\\9\\-2\end{pmatrix} \cdot \begin{pmatrix}18\\-16\\-10\end{pmatrix}$$

$$r \cdot \begin{pmatrix}18\\-16\\-10\end{pmatrix} = 11\times18 + 9\times-16 + -2\times-10 = 74 \quad ①$$

To convert to Cartesian, replace r with $\begin{pmatrix}x\\y\\z\end{pmatrix}$:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}18\\-16\\-10\end{pmatrix} = 74 \implies 18x - 16y - 10z = 74 \quad ①$$

Question 9 continued

(ii) $\cos^2 60 + \cos^2 45 + \cos^2\theta = 1$

$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \cos^2\theta = 1$ ①

$\cos^2\theta = 1 - \frac{1}{4} - \frac{2}{4}$

$\cos\theta = \sqrt{\frac{1}{4}}$

$\cos\theta = \pm\frac{1}{2}$ ①

$\frac{x}{\cos 60} = \frac{y}{\cos 45} = \frac{z}{\cos\theta}$ ← vector equation of a 3D line

$\frac{x}{\frac{1}{2}} = \frac{y}{\frac{\sqrt{2}}{2}} = \frac{z}{\pm\frac{1}{2}}$ ①

$2x = \sqrt{2}y = 2z$ and $2x = \sqrt{2}y = -2z$ ①

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Question 9 continued

(Total for Question 9 is 10 marks)

10. The motion of a particle P along the x-axis is modelled by the differential equation

$$t^2\frac{d^2x}{dt^2} - 2t(t+1)\frac{dx}{dt} + 2(t+1)x = 8t^3e^t \qquad \text{(I)}$$

where P has displacement x metres from the origin O at time t minutes, $t > 0$

(a) Show that the transformation $x = tu$ transforms the differential equation (I) into the differential equation

$$\frac{d^2u}{dt^2} - 2\frac{du}{dt} = 8e^t$$

(4)

Given that P is at O when $t = \ln 3$ and when $t = \ln 5$

(b) determine the particular solution of the differential equation (I)

(8)

(a) Let $x = tu$

product rule

$$\frac{dx}{dt} = u\frac{d}{dt}(t) + t\frac{d}{dt}(u)$$

$$\frac{dx}{dt} = u + t\frac{du}{dt} \quad 1$$

product rule

$$\frac{d^2x}{dt^2} = \frac{du}{dt} + \left[\frac{du}{dt} + t\frac{d}{dt}\left(\frac{du}{dt}\right)\right]$$

$$\frac{d^2x}{dt^2} = 2\frac{du}{dt} + t\frac{d^2u}{dt^2} \quad 1$$

sub back into ODE:

$$t^2\left(2\frac{du}{dt} + t\frac{d^2u}{dt^2}\right) - 2t(t+1)\left(u + t\frac{du}{dt}\right)$$

$$+ 2(t+1)(tu) = 8t^3e^t$$

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Question 10 continued

$$2t^2\frac{du}{dt} + t^3\frac{d^2u}{dt^2} - (2t^2 + 2t)\left(u + t\frac{du}{dt}\right)$$
$$+ 2t^2u + 2tu = 8t^3e^t$$

$$t^3\frac{d^2u}{dt^2} - 2t^3\frac{du}{dt} = 8t^3e^t \quad 1$$

$$\frac{d^2u}{dt^2} - 2\frac{du}{dt} = 8e^t \quad 1$$

b) Aux $\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0, \quad \lambda = 2$ 1

CF. $u = A + Be^{2t}$ 1

PI: $u = ae^t$ 1

$\dot{u} = ae^t$

$\ddot{u} = ae^t$

sub into ODE.

$ae^t - 2ae^t = 8e^t$

$-a = 8$

$a = -8$ 1

Question 10 continued

CF . $u = A + Be^{2t}$

PI . $u = -8e^{t}$

GS . $u = A + Be^{2t} - 8e^{t}$ ①

Recall that $x = tu$ and $u = \frac{x}{t}$

$\frac{x}{t} = A + Be^{2t} - 8e^{t}$

$x = t(A + Be^{2t} - 8e^{t})$

When $x = 0$, $t = \ln 3$, $\ln 5$

$0 = \ln 3(A + Be^{2\ln 3} - 8e^{\ln 3})$

$0 = A + 9B - 24$

$0 = \ln 5(A + B^{2\ln 5} - 8e^{\ln 5})$

$0 = A + 25B - 40$ ①

$A = 15$, $B = 1$ ①

$\therefore x = (15 + e^{2t} - 8e^{t})t$ ①

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Question 10 continued

Question 10 continued

(Total for Question 10 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

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